Odd and Even Functions

Before we introduce hyperbolic functions, we need to understand odd and even functions.

Fact (Even and Odd Functions) — A function $f(x)$ is:

- **even** if $f(-x) = f(x)$ for all x in its domain
- **odd** if $f(-x) = -f(x)$ for all x in its domain

Example

$\sin(x)$ is an odd function

$$\sin(-x) \equiv -\sin(x)$$

Theorem (Decomposition into Odd and Even Functions)

Any function f with a symmetric domain can be written as the sum of an even function g and an odd function h:

$$f(x) = g(x) + h(x)$$

Proof:

First, verify that g is even:

$$g(-x) = \frac{f(-x) + f(x)}{2} = \frac{f(x) + f(-x)}{2} = g(x)$$

Next, verify that h is odd:

$$h(-x) = \frac{f(-x) - f(x)}{2} = -\frac{f(x) - f(-x)}{2} = -h(x)$$

Finally, check that $g(x) + h(x) = f(x)$:

$$g(x) + h(x) = \frac{f(x) + f(-x)}{2} + \frac{f(x) - f(-x)}{2} = f(x)$$

Example

Express $f(x) = e^x$ as the sum of an even and an odd function.

Using the theorem:

$$g(x) = \frac{e^x + e^{-x}}{2} \quad \textit{(even function)}$$
$$h(x) = \frac{e^x - e^{-x}}{2} \quad \textit{(odd function)}$$

These turn out to be so important that they have special names!

Hyperbolic Function Definitions

Fact (Hyperbolic Sine and Cosine) — The hyperbolic functions are defined as:

$$\cosh x = \frac{e^x + e^{-x}}{2} \quad \text{and} \quad \sinh x = \frac{e^x - e^{-x}}{2}$$

Note that $\cosh x$ is even and $\sinh x$ is odd.

Tip

The notation comes from adding 'h' (for 'hyperbolic') to the trigonometric symbols. Pronounce sinh as "shine", and cosh as "cosh".

Example

Evaluate (a) $\cosh(0)$, (b) $\sinh(0)$, (c) $\cosh(\ln 2)$, (d) $\sinh(\ln 3)$

(a) $\cosh(0) = \frac{e^0+e^0}{2} = \frac{1+1}{2} = 1$

(b) $\sinh(0) = \frac{e^0-e^0}{2} = \frac{1-1}{2} = 0$

(c) $\cosh(\ln 2) = \frac{e^{\ln 2}+e^{-\ln 2}}{2} = \frac{2+\frac{1}{2}}{2} = \frac{5}{4}$

(d) $\sinh(\ln 3) = \frac{e^{\ln 3}-e^{-\ln 3}}{2} = \frac{3-\frac{1}{3}}{2} = \frac{4}{3}$

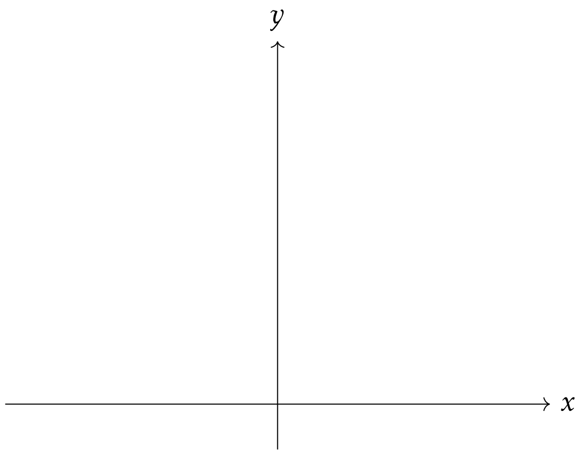
Example

What is $\tanh x$?

Graphs of Hyperbolic Functions

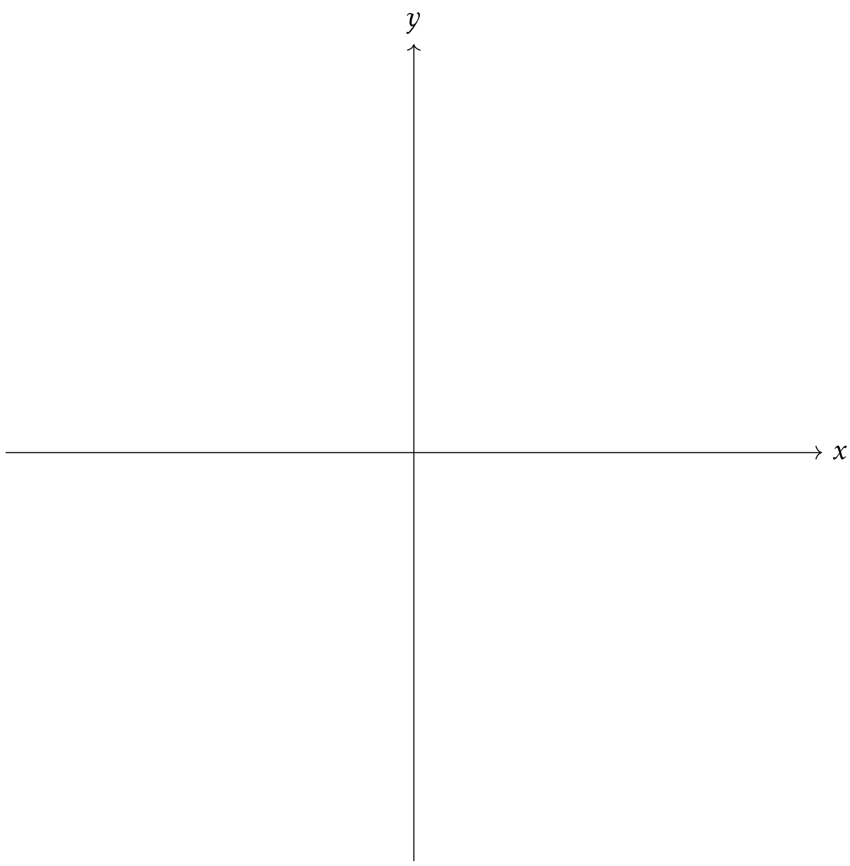
Example

Sketch $y = \cosh x$



Example

Sketch $y = \sinh x$

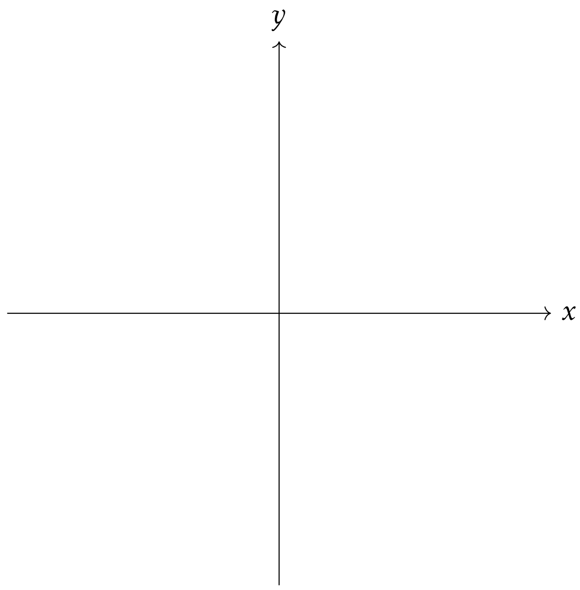


Fact (Key Properties of cosh, sinh and tanh x) —

- $\cosh x \geq 1$ for all x, with minimum at $x = 0$
- $\sinh x, \tanh x$ passes through the origin and is strictly increasing
- As $x \to \infty$: $\cosh x \approx \sinh x \approx \frac{1}{2}e^{x}$, $\tanh x \approx 1$
- As $x \to -\infty$: $\cosh x \approx \frac{1}{2}e^{-x}$, $\sinh x \approx -\frac{1}{2}e^{-x}$, $\tanh x \approx -1$

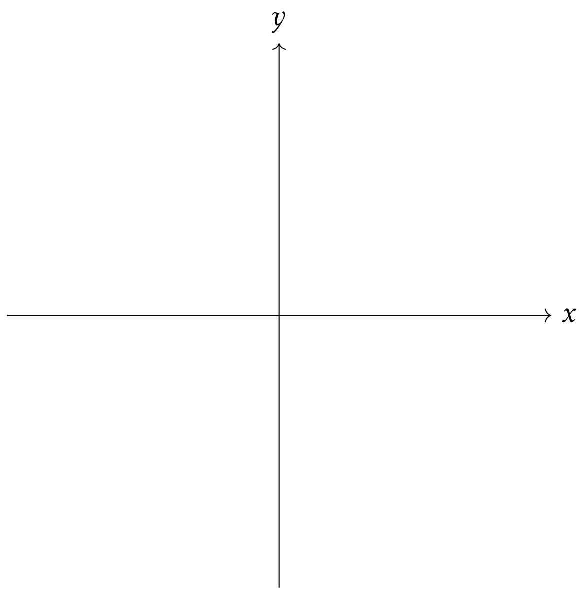
Example

Sketch $y = 2\sinh(x+2)$



Example

Sketch $y = \operatorname{sech}(x)$



Hyperbolic Identities

Example

Prove that $\cosh^2 x - \sinh^2 x \equiv 1$

$$\begin{aligned}\cosh^2 x - \sinh^2 x &\equiv \left(\frac{e^x + e^{-x}}{2}\right)^2 - \left(\frac{e^x - e^{-x}}{2}\right)^2 \\ &\equiv \frac{e^{2x} + 2 + e^{-2x}}{4} - \frac{e^{2x} - 2 + e^{-2x}}{4} \\ &\equiv \frac{2}{4} + \frac{2}{4} \\ &\equiv 1\end{aligned}$$

Tip

Compare with $\cos^2 x + \sin^2 x = 1$. Note the sign difference!

Fact (Addition Formulae) —

$$\begin{aligned}\sinh(A+B) &= \sinh A \cosh B + \cosh A \sinh B \\ \sinh(A-B) &= \sinh A \cosh B - \cosh A \sinh B \\ \cosh(A+B) &= \underbrace{\cosh A \cosh B + \sinh A \sinh B}_{\text{notice the sign difference}} \\ \cosh(A-B) &= \underbrace{\cosh A \cosh B - \sinh A \sinh B}_{\text{notice the sign difference}}\end{aligned}$$

Example

Prove that $\sinh(A+B) = \sinh A \cosh B + \cosh A \sinh B$

Using the exponential definitions:

$$\begin{aligned}\sinh(A+B) &= \frac{e^{A+B} - e^{-(A+B)}}{2} \\ &= \frac{e^A e^B - e^{-A} e^{-B}}{2}\end{aligned}$$

Now:

$$\begin{aligned}\sinh A \cosh B + \cosh A \sinh B &= \frac{e^A - e^{-A}}{2} \cdot \frac{e^B + e^{-B}}{2} + \frac{e^A + e^{-A}}{2} \cdot \frac{e^B - e^{-B}}{2} \\ &= \frac{1}{4}[(e^A - e^{-A})(e^B + e^{-B}) + (e^A + e^{-A})(e^B - e^{-B})]\end{aligned}$$

$$= \frac{1}{4}[2e^A e^B - 2e^{-A} e^{-B}]$$
$$= \frac{e^A e^B - e^{-A} e^{-B}}{2} = \sinh(A+B)$$

Fact (Double Angle Formulae) —

$$\cosh 2A = 2\cosh^2 A - 1 = \underbrace{\cosh^2 A + \sinh^2 A}_{\text{notice the sign difference}} = \underbrace{1 + 2\sinh^2 A}_{\text{notice the sign difference}}$$
$$\sinh 2A = 2\sinh A \cosh A$$

Tip (Osborn's Rule)

When converting a trignometric identity to a hyperbolic trig identity, "stick an 'h' on the end" and whenever two sins are multiplied together, flip the sign

Example

Prove that $\cosh^4 x \equiv \frac{1}{8}\cosh 4x + \frac{1}{2}\cosh 2x + \frac{3}{8}$

$$\begin{aligned}
\cosh^4 x &\equiv \left(\frac{e^x + e^{-x}}{2}\right)^4 \\
&\equiv \frac{1}{16}\left(e^{4x} + \binom{4}{1}e^{3x}e^{-x} + \binom{4}{2}e^{2x}e^{-2x} + \binom{4}{3}e^{x}e^{-3x} + e^{-4x}\right) \\
&\equiv \frac{1}{16}\left(e^{4x} + 4e^{2x} + 6 + 4e^{-2x} + e^{-4x}\right) \\
&\equiv \frac{1}{16}\left(e^{4x} + e^{-4x} + 4(e^{2x} + e^{-2x}) + 6\right) \\
&\equiv \frac{1}{8}\cosh 4x + \frac{1}{2}\cosh 2x + \frac{3}{8}
\end{aligned}$$

Example

Prove that $\sinh 5x \equiv 16\sinh^5 x + 20\sinh^3 x + 5\sinh x$

$$\begin{aligned}
\sinh 5x &\equiv \frac{1}{2}\left(e^{5x} - e^{-5x}\right) \\
&\equiv \frac{1}{2}\left((\cosh x + \sinh x)^5 - (\cosh x - \sinh x)^5\right) \\
&\equiv \frac{1}{2}\Big(\left(\cosh^5 x + 5\cosh^4 x\sinh x + 10\cosh^3 x\sinh^2 x + 10\cosh^2 x\sinh^3 x + 5\cosh x\sinh^4 x + \sinh^5 x\right) + \cdots \\
&\qquad \cdots - \left(\cosh^5 x - 5\cosh^4 x\sinh x + 10\cosh^3 x\sinh^2 x - 10\cosh^2 x\sinh^3 x + 5\cosh x\sinh^4 x - \sinh^5 x\right)\Big) \\
&\equiv 5\cosh^4 x\sinh x + 10\cosh^2 x\sinh^3 x + \sinh^5 x \\
&\equiv 5(1 + \sinh^2 x)^2\sinh x + 10(1 + \sinh^2 x)\sinh^3 x + \sinh^5 x \\
&\equiv 5\sinh x + (10 + 10)\sinh^3 x + (5 + 10 + 1)\sinh^5 x \\
&\equiv 16\sinh^5 x + 20\sinh^3 x + 5\sinh x
\end{aligned}$$

Differentiating hyperbolic trig functions

Example

Find $\frac{d}{dx}(\cosh x)$ and $\frac{d}{dx}(\sinh x)$

Example

Find $\frac{d}{dx}(\tanh x)$

Maclaurin Series

Fact (Maclaurin Series for e^x) — From $e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \cdots$

Example

Find the Maclaurin series for $\cosh x$ and $\sinh x$

Tip

Note that $\cosh x$ has only even powers (it's an even function) and $\sinh x$ has only odd powers (it's an odd function). All coefficients are positive!

Example

Find the Maclaurin series for $\tanh x$ up to the x^5 term.

Method 1: Direct differentiation
Let $f(x) = \tanh x$. *Then:*

$$\begin{aligned} f(0) &= 0 \\ f'(x) &= sech^2 x, \quad f'(0) = 1 \\ f''(x) &= -2sech^2 x \tanh x, \quad f''(0) = 0 \\ f'''(x) &= -2sech^2 x \cdot sech^2 x + 2sech^2 x \cdot 2sech^2 x \tanh^2 x \\ &= -2sech^4 x(1 - 2\tanh^2 x), \quad f'''(0) = -2 \\ f^{(4)}(0) &= 0, \quad f^{(5)}(0) = 16 \end{aligned}$$

Therefore: $\tanh x = x - \frac{x^3}{3} + \frac{2x^5}{15} + \cdots$
Method 2: Since $\tanh x$ *is odd, we expect only odd powers:*

$$\tanh x = x - \frac{x^3}{3} + \frac{2x^5}{15} - \cdots \quad \textit{for } |x| < \frac{\pi}{2}$$

Inverse Hyperbolic Functions

Example

Find a formula for $\sinh^{-1} x$

Example

Find a formula for $\tanh^{-1} x$